Model \textit{CreditRisk+}: The Economic Perspective of Portfolio Credit Risk
Part II

Seminar: Portfolio Credit Risk
Instructor: Rafael Weissbach
Speaker: Kexun Li
Agenda

Review to the modeling assumptions and calibration
Sector analysis
Active Portfolio Management
What did we do last week?

- Closed-form analytic model
- Only probability of default (PD) is stochastic
- Each obligor has a *definite known* PD over one year: $p_A$
- Default rate of each obligor follows the Bernoulli distribution $X_A \sim B(1, p_A)$
- Small default rates of obligors
Review to the modeling assumptions and calibration

What did we do last week?

- Probability generating function (pgf) for the expected number of default events

\[
F(z) = \prod_A F_A(z) = \prod_A (1 + p_A(z - 1))
\]

\[
F(z) = e^{\mu(z-1)} = e^{-\mu}e^{\mu z} = \sum_{n=0}^{\infty} \frac{e^{-\mu} \mu^n}{n!} z^n \quad \Rightarrow \quad \Pr(\text{n defaults}) = \frac{e^{-\mu} \mu^n}{n!}
\]

- The pgf for the portfolio losses distribution

\[
G_j(z) = \sum_{n=0}^{\infty} p(\text{n defaults}) z^{n \nu_j} = \sum_{n=0}^{\infty} \frac{e^{-\mu_j \nu} \mu_j^n}{n!} z^{n \nu_j} = e^{-\mu_j + \mu_j z^{\nu_j}}
\]

\[
G(z) = e^{\mu(P(z)-1)} = F(P(z))
\]
Agenda

Review to the modeling assumptions and calibration  
**Sector analysis**  
Active Portfolio Management
Idea of sector analysis:
• PDs are volatile over time
• A small number of background factors (systematic factors, concentration risk) as driver of variability of PDs
• Allocating all obligors into a single sector, in which PDs move together
• Sectors (background factors) are mutually independent
• Specific factors (idiosyncratic factors) capture uniquely the fortune of an obligor

Basic Modeling:
• Allocating each obligor into a single sector
• General sector analysis: variability of each obligor’s PD is simultaneously influenced by several background factors
## Sector analysis: Notation

<table>
<thead>
<tr>
<th>Obligor (A)</th>
<th>Sector $S_k$ for $1 \leq k \leq n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Random variable of number of defaults</td>
<td>$X_A$, $X_k$</td>
</tr>
<tr>
<td>Mean value</td>
<td>$\mu_A$, $\mu_k$</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>$\sigma_A$, $\sigma_k$</td>
</tr>
<tr>
<td>Probability of default</td>
<td>$p_A$, $p_k$</td>
</tr>
<tr>
<td>Basic unit of exposure</td>
<td>$L$</td>
</tr>
<tr>
<td>Exposure sizes in units</td>
<td>$L_A = L \nu_A$, $L^{(k)} = L \nu^{(k)}_j$ for $1 \leq k \leq n$, $1 \leq j \leq m(k)$</td>
</tr>
<tr>
<td>Expected Losses in units</td>
<td>$\lambda_A = L \epsilon_A$, $\lambda^{(k)}_j = L \epsilon^{(k)}_j$ for $1 \leq k \leq n$, $1 \leq j \leq m(k)$</td>
</tr>
</tbody>
</table>
Properties of pdf within one sector

Additivity of the first and the second moments:

- mean value of one sector: \( \mu_k = \sum \frac{\varepsilon_A}{\nu_A} = \sum p_A \)

- standard deviation (SD): \( \sum \sigma_A = \sum \frac{\varepsilon_A \sigma_A}{\nu_A \mu_A} = \sigma_k \frac{1}{\mu_k} \sum \frac{\varepsilon_A}{\nu_A} = \sigma_k \)

Linear combination between mean and SD:

\[
\frac{\sigma_k}{\mu_k} = \frac{\sum \sigma_A}{\sum p_A} = \frac{\sum p_A \left( \frac{\sigma_A}{p_A} \right)}{\sum p_A} = \frac{\sum p_A \omega_k}{\sum p_A}
\]

the ratio \( \frac{\sigma_A}{p_A} \) is according to historical data the same for obligors in the same sector: \( \omega_k \)

The SD can be generated by estimating the ratio \( \omega_k \)

\( \sigma_k = \omega_k \times \mu_k \)
Excursus: Gamma Distribution I

Written \( \Gamma(\alpha, \beta) \)

Continous two parameter distribution with the pdf:

\[
P(x \leq X \leq x + dx) = \frac{1}{\beta^\alpha \Gamma(\alpha)} e^{-\frac{x}{\beta}} x^{\alpha - 1} dx
\]

\[
\mu = \alpha \beta \quad \sigma^2 = \alpha \beta^2
\]

In our sector model:

\[
\alpha_k = \frac{\mu_k^2}{\sigma_k^2}
\]

\[
\beta_k = \frac{\sigma_k^2}{\mu_k}
\]

Why gamma distribution:

- Actuarial method
- Skewness: the smaller the alpha, the closer the distribution to empirical data
- Mixture form of Poisson rated gamma distribution (later)
Excursus: Gamma Distribution II

- $\alpha = 1$, $\beta = 2.0$
- $\alpha = 2$, $\beta = 2.0$
- $\alpha = 3$, $\beta = 2.0$
- $\alpha = 5$, $\beta = 0.1$
- $\alpha = 9$, $\beta = 0.5$

Sector analysis
Excursus: Negative Binomial Distribution I

Discrete two parameter distribution

Describing the probability distribution of $n$ defaults and $\alpha$ survivals in a series of i.i.d. Bernoulli trials with survival on the last trial.

Probability of defaults: $p$  
Probability of survival: $1-p$

$p$mf:

$$\left(1 - p\right)^\alpha \binom{n + \alpha - 1}{n} p^n$$

$pgf$:

$$g_x(z) = \left(\frac{1 - p}{1 - pz}\right)^\alpha$$

Why negative binomial distribution

- The most important distribution in actuary beside Poisson distribution
- Skewness
- Appropriate to recurrence computation (in order to generate portfolio losses distribution)
Excursus: Negative Binomial Distribution II

Negative binomial distribution:
- target successes = 2
- probability of default = 0.1

Number of defaults until target successes

Sector analysis
Key assumption:
Sector default rates are Gamma distributed with mean $\mu_k$ and standard deviation $\sigma_k$

$$X_k \sim \Gamma(\alpha_k, \beta_k) \quad \text{with} \quad \alpha_k = \frac{\mu_k^2}{\sigma_k^2} \quad \text{and} \quad \beta_k = \frac{\sigma_k^2}{\mu_k}$$

pgf of conditional default rate:

$$F_k(z)[X_k = x] = e^{x(z-1)}$$

pgf of unconditional default rate:

$$F_k(z) = \sum_{n=0}^{\infty} P(n\text{defaults})z^n = \sum_{n=0}^{\infty} z^n \int_{x=0}^{\infty} P(n\text{defaults}|x)f(x)dx$$

$$= \int_{x=0}^{\infty} e^{x(z-1)} f(x)dx$$

Poisson characteristic
Gamma distribution
Mixture model
Sector analysis

**Modeling with stochastic PD: default rate distribution II**

\[
F_k(z) = \int_{x=0}^{\infty} e^{x(z-1)} \frac{x^\beta x^{\alpha-1}}{\beta^\alpha \Gamma(\alpha)} \, dx = \ldots = \frac{1}{\beta^\alpha (1+\beta^{-1}z)^\alpha}
\]

pgf for single sector k:

\[
F_k(z) = \left(\frac{1-p_k}{1-p_k z}\right)^{\alpha_k}
\]

where \( p_k = \beta_k / (1 + \beta_k) \)

Expanding in Taylor series:

\[
F_k(z) = (1-p_k)^{\alpha_k} \sum_{n=1}^{\infty} \binom{n + \alpha_k - 1}{n} p_k^n z^n
\]

\( \rightarrow \) negative binomial distribution
Modeling with stochastic PD: losses distribution

Capture portfolio losses in pgf: using exposure multiples $v_A$

Pgf of portfolio losses distribution:

$$G_k(z) = \sum_{n=0}^{\infty} z^n \int_{x_k=0}^{\infty} P(\text{Loss of } nL|x_k) f_k(x_k) \, dx_k = \int_{x_k=0}^{\infty} e^{\sum_{A} x_A (z^{v_A} - 1)} f_k(x_k) \, dx_k$$

$$G(z) = \prod_{k=1}^{n} G_k(z) = \prod_{k=1}^{n} \left( \frac{1 - p_k}{1 - \frac{p_k}{\mu_k} \sum_{j=1}^{m(k)} \frac{\epsilon_j^{(k)}}{v_j^{(k)} z^{v_j^{(k)}}}} \right)^{\alpha_k}$$

$$p_k = \frac{\beta_k}{1 + \beta_k}$$

Easy calculation through general recurrence relation

Example spreadsheet-based implementation
Agenda

Review to the modelling assumptions and calibration
Sector analysis
Active portfolio management
  • General sector analysis
  • Pairwise correlation
  • Risk contribution
Active portfolio management

Effect of concentration and correlation on credit risk

1. Concentration of portfolio
2. Correlation of borrower behaviour

Diversification of credit risk

Credit risk

Specific risk: driven by concentration
Systematic risk: driven by correlation

Size of portfolio
Potential benefits in a typical bank portfolio

Source: Oliver, Wyman & Company (1999): Credit portfolio management
Agenda

Review to the modelling assumptions and calibration
Sector analysis
Active portfolio management
  • General sector analysis
  • Pairwise correlation
  • Risk contribution
General sector analysis I

Idea of general sector analysis
• Replacing the concept of a sector with that of a systematic factor
• Factor analysis: “factor loading”
• Sensitivity of every background factor to a single obligor $\theta_{Ak}$
• Critical assumption of CR+ : independent background factors

Starting from the pgf of losses distribution

$$G_k(z) = \int_{x_k=0}^{\infty} e^{z \sum_{A} x_A (z_A -1)} f_k(x_k) dx_k$$
Active portfolio management

General Sector Analysis II

Previously:
\[ \sum_{A} x_A (z^{v_A} - 1) = \frac{x_k}{\mu_k} \sum_{A} \frac{\varepsilon_A}{v_A} (z^{v_A} - 1) \]

Now:
\[ \sum_{A} x_A (z^{v_A} - 1) = \sum_{A} \theta_{Ak} \frac{x_k}{\mu_k} \frac{\varepsilon_A}{v_A} (z^{v_A} - 1) \]

Then
\[ x_A = \frac{\varepsilon_A}{v_A} \sum_{k=1}^{n} \theta_{Ak} \frac{x_k}{\mu_k} \]

- \( \theta_{Ak} \) explains the influence of one systematic factor \( k \) to a single obligor \( A \)
- Sum of all influences to an obligor is equal to 1

For a sector \( k \):
\[ \mu_k = \sum_{A} \theta_{Ak} \frac{\varepsilon_A}{v_A} \quad \sigma_k = \sum_{A} \theta_{Ak} \sigma_A \]

Example spreadsheet-based implementation
Agenda

Review to the modelling assumptions and calibration
Sector analysis
Active portfolio management
• General sector analysis
• Pairwise correlation
• Risk contribution
Pairwise correlation

Define a indicator function:

\[ I_A = \begin{cases} 
1 & \text{if obligor A defaults in the time period} \\
0 & \text{otherwise} 
\end{cases} \]

Correlation coefficient:

\[ \rho_{AB} = \rho(I_A, I_B) \]

Default event correlation between distinct obligors A and B:

\[ \rho_{AB} = \sqrt{\mu_A \mu_B} \sum_{k=1}^{n} \theta_A \theta_B \left( \frac{\sigma_k}{\mu_k} \right)^2 \]

- If the obligors A and B have no sector in common, then the correlation between them will be zero (No systematic factor affects them)
- In general one would expect default correlations to typically be of the same order of magnitude as default probabilities themselves. (order: \( \sqrt{\mu_A \mu_B} \))
Active portfolio management

**Correlation (extension)**
what happens if sectors are mutually dependant?

Version I: Bürgisser et al’s model
  Integrating correlation

Version II: Factor loading through principal axis transformation

Version III: Global background factor

Underestimation of default losses from the model *CreditRisk+*

![Graph showing probability distribution](Source: Buergisser (1999))

![Graph showing tail probability distribution](Source: Giese(1999))

Oct. 4<sup>th</sup> 2007
Agenda

Review to the modelling assumptions and calibration
Sector analysis
Active portfolio management
  •  General sector analysis
  •  Pairwise correlation
  •  Risk contribution
Risk contribution and capital allocation

Key problem: who drives the risk and how much?

- Ex ante marginal risk contribution
- Serving for risk-based pricing
- From risk measurement to risk management
- Risk contribution in CR+: analytic approximation without simulation
Active portfolio management

Risk Contribution

Definitions of Risk Contribution:
- Marginal effect of a single obligor on the standard deviation of the distribution of credit losses (UL)
  \[ RC_A = E_A \frac{\partial \sigma}{\partial E_A} \]
  \[ RC_A = \frac{E_A \frac{\partial \sigma^2}{\partial E_A}}{2\sigma \frac{\partial E_A}{\partial E_A}} \]
- Marginal effect of a single obligor on a given loss percentile of portfolio aggregate risk (VaR)
  \[ \text{VaR}_A (p_{\text{VaR}}) = \epsilon_A + \xi RC_A \]
  \[ P(L \leq E[L] + \xi \sqrt{\text{Var}(L)}) = 1 - p_{\text{VaR}} \]
Active portfolio management

Calibrating risk contribution on standard deviation

Variance of loss distribution for the whole portfolio:

$$\sigma^2 = \sum_{k=1}^{n} \varepsilon_k^2 \left( \frac{\sigma_k}{\mu_k} \right)^2 + \sum_A \varepsilon_A V_A$$

RC for obligor A on standard deviation:

$$RC_A = E_A \frac{\partial \sigma}{\partial E_A} = \frac{E_A}{2\sigma} \frac{\partial \sigma^2}{\partial E_A}$$

$$RC_A = \frac{E_A \mu_A}{\sigma} \left( E_A + \sum_k \left( \frac{\sigma_k}{\mu_k} \right)^2 \varepsilon_k \theta_Ak \right)$$
Portfolio management using Risk Contribution

- The total risk contributions for the individual obligors is approximately equal to the risk of the entire portfolio

\[ \sum_A RC_A = \frac{1}{2\sigma} \sum A E_A \frac{\partial \sigma^2}{\partial E_A} = \frac{2\sigma^2}{2\sigma} = \sigma \]

- Risk contributions allow the effect of a potential change in the portfolio (e.g. the removal of an exposure) to be measured
- In general, a portfolio can be effectively managed by focusing on a relatively few obligors that represent a significant proportion of the risk but constitute a relatively small proportion of the absolute portfolio exposures

Example spreadsheet-based implementation
Summary:

Sector analysis
Modeling the loss distribution through probability generating function, where the total probability of defaults for n sectors follows a gamma distribution, default rate of obligors in single sector is Poisson-distributed and the sectors are independent to each other.

Active portfolio management using CreditRisk+
General sector analysis allows the portfolio to be allocated to sectors to reflect the degree of diversification and concentration.

Pairwise correlation represents a analytical form of portfolio management in the level of single obligors

Risk contribution is widely used to reduce Incremental Credit Reserve
Discussion:

Advantages and disadvantages of CreditRisk+ ?
Literatures


Oliver, Wyman & Company (1999): Credit portfolio management, ERisk.com